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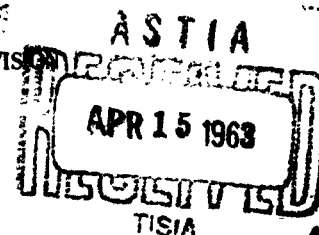
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## Thermoelastic Vibrations of a Beam

20 FEBRUARY 63

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UNITED STATES AIR FORCE  
Norton Air Force Base, California*



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LABORATORIES DIVISION • AEROSPACE CORPORATION  
CONTRACT NO. AF 04(695)-169

**BSD-TDR-63-17**

**Report No.  
TDR-169(3153-06)TR-1**

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## ABSTRACT

The equations of motion of a beam with the lateral surface thermally insulated are derived, including the effects of shear deformation and rotatory inertia. Thermoelastic coupling in the heat conduction equation and in the elastic constitutive relations is also included. Axial deformations of the beam are taken into account. Two examples are presented and discussed.

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## SYMBOLS

$A$	cross-sectional area
$C$	boundary of cross section of a beam
$c$	specific heat at constant volume
$c_1^2$	$E/\rho$
$E$	modulus of elasticity
$e$	$\bar{u},_x + \bar{v},_y + \bar{w},_z$
$F(t), G(t)$	see pages 18 and 20
$I(\omega)$	imaginary part of $\omega$
$I_{xx}, I_{yy}, I_{xy}$	moments of inertia
$K_x, K_y$	Timoshenko shear constants
$k$	thermal conductivity
$M_x, M_y$	bending moments (see page 5)
$N$	axial force (see page 5)
$n$	mode number
$P_x, P_y, P_z$	applied forces per unit length (see page 5)
$Q_x, Q_y$	shear resultants (see page 5)
$R_x, R_y$	applied moments per unit length (see page 5)
$T$	$\int_A \bar{T} dA$
$\bar{T}$	temperature measured from ambient
$T_0$	ambient temperature (absolute)
$T_x, T_y$	$\int_A y, x \bar{T} dA$
$U, V, W, W_x, W_y$	displacement resultants (see page 5)
$u, v, w$	$1/A \int_A \bar{u}, \bar{v}, \bar{w} dA$
$\bar{u}, \bar{v}, \bar{w}$	displacements

$x, y, z$	rectangular Cartesian coordinates
$\alpha$	coefficient of thermal expansion
$\gamma$	$l c_1 / (\kappa n \pi)$ (see page 19), $\left[ n^2 \pi^2 / (\kappa l^2) [E I_{yy} / (\rho A)]^{1/2} \right] / (A / I_{yy} + n^2 \pi^2 / l^2)$ (see page 22)
$\epsilon$	$\left[ (3\lambda + 2\mu) \alpha^2 T_0 \right] / [(\lambda + \mu) \rho c]$
$\zeta$	$\omega l / (n \pi c_1)$ (see page 19), $\omega l^2 / (n^2 \pi^2) [\rho A / (E I_{yy})]^{1/2}$ (see page 22)
$\zeta_1, \zeta_2, \zeta_3$	roots of the frequency equations, Eqs. (39) and (49)
$\nu$	Poisson's ratio
$\iota$	$(-1)^{1/2}$
$\kappa$	$k / \rho c$ thermal diffusivity
$\lambda, \mu$	Lamé elastic constants
$\rho$	density
$\sigma_{xx}, \sigma_{yy}, \text{ etc.}$	stress components
$\phi_x, \phi_y$	cross-sectional rotations (see page 13)
$\omega$	frequency

## SECTION I

### INTRODUCTION

In treating the problem of calculating stresses and deflections of a complicated structure, the common practice is to consider the structure as being made of a number of simpler structural elements whenever possible and to treat these elements separately. Of all of the simple structural elements, the beam is perhaps the most common and at the same time the most useful element. In many applications, the beam can be thought of as having dynamic tractions and temperature gradients applied to its bounding surfaces. This report treats the problem of the dynamic response of a beam to dynamic loads and to thermal gradients applied to the ends of the beam under the condition that the lateral surface of the beam is insulated. This latter condition could be fulfilled approximately if the beam were in the hard vacuum of a space environment and if the thermal radiation from its lateral surfaces could be neglected.

It is well known<sup>1, 2</sup> that the temperature couples with the elastic problem in two ways: the coupling in the constitutive relations described by Neumann<sup>3</sup> and the coupling in the heat conduction equation described by Biot.<sup>1</sup> This latter coupling can be neglected for a wide class of problems,<sup>2</sup> and several articles have treated the problems associated with this assumption.<sup>4-6</sup> Ignaczak and Nowacki<sup>7</sup> have presented a similar analysis; however, because the reference was not available to the author, no appraisal of it relative to the present work could be made. The Biot coupling is included in this report in a general theory which also includes rotatory inertia and shear deformation.

## SECTION II

### ANALYSIS OF RAYLEIGH BEAM

The coupled thermoelastic equations of motion in a rectangular Cartesian coordinate system are<sup>1, 2</sup>

$$\left. \begin{aligned} \sigma_{xx, x} + \sigma_{xy, y} + \sigma_{xz, z} &= \rho \bar{u}_{, tt} \\ \sigma_{xy, x} + \sigma_{yy, y} + \sigma_{yz, z} &= \rho \bar{v}_{, tt} \\ \sigma_{xz, x} + \sigma_{yz, y} + \sigma_{zz, z} &= \rho \bar{w}_{, tt} \end{aligned} \right\} \quad (1)$$

$$k \nabla^2 \bar{T} = \rho c \bar{T}_{, t} + (3\lambda + 2\mu) \alpha T_0 e_{, t} \quad (2)$$

$$\left. \begin{aligned} \sigma_{xx} &= \lambda e + 2\mu \bar{u}_{, x} - (3\lambda + 2\mu) \alpha \bar{T} \\ \sigma_{yy} &= \lambda e + 2\mu \bar{v}_{, y} - (3\lambda + 2\mu) \alpha \bar{T} \\ \sigma_{zz} &= \lambda e + 2\mu \bar{w}_{, z} - (3\lambda + 2\mu) \alpha \bar{T} \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \sigma_{xy} &= \mu (\bar{u}_{, y} + \bar{v}_{, x}) \\ \sigma_{xz} &= \mu (\bar{u}_{, z} + \bar{w}_{, x}) \\ \sigma_{yz} &= \mu (\bar{v}_{, z} + \bar{w}_{, y}) \end{aligned} \right\} \quad (4)$$

where  $\sigma_{xx}$ ,  $\sigma_{yy}$ , etc., are the usual components of the stress tensor;  $\lambda$  and  $\mu$  are the Lamé constants;  $\bar{u}$ ,  $\bar{v}$ ,  $\bar{w}$  are the displacement components;  $k$  is the

thermal conductivity;  $\rho$  is the density;  $\bar{T}$  is the temperature;  $c$  is the specific heat of constant volume;  $e$  is the dilatation and is given by  $\bar{u}_{,x} + \bar{v}_{,y} + \bar{w}_{,z}$ . A comma followed by a subscript denotes differentiation.

The terms  $(3\lambda + 2\mu)\alpha \bar{T}$  occurring in Eq. (3) are the usual Neumann coupling terms.<sup>3, 8</sup> The term  $(3\lambda + 2\mu)\alpha T_0 e_{,t}$  in Eq. (3) is the coupling described by Biot.<sup>1</sup> The method of procedure is to integrate Eqs. (1) through (4) across the thickness of the beam. The centroid of the cross section of the beam lies along the  $z$  axis, as shown in Fig. 1. Green's lemma is used in the form

$$\int_A (P_{,x} + Q_{,y}) dA = \int_C -Q dx + P dy,$$

where  $A$  denotes the cross-sectional area of the beam and  $C$  the boundary of the cross section. The equations will be derived first neglecting the shear deformation of the beam which will be put in later.

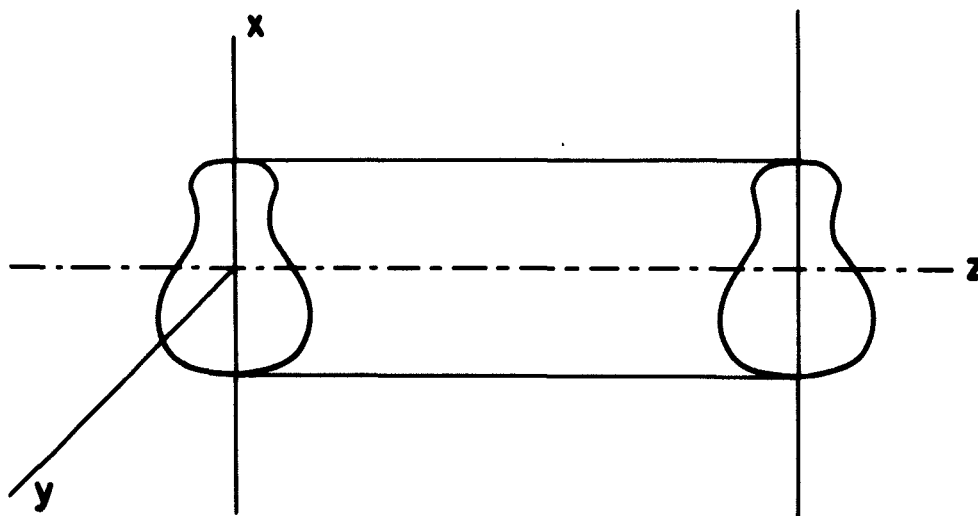


Fig. 1. Beam geometry.

Define:

$$\left. \begin{aligned}
 Q_x &= \int_A \sigma_{xz} dA, & Q_y &= \int_A \sigma_{yz} dA \\
 N &= \int_A \sigma_{zz} dA, & M_y &= \int_A x \sigma_{zz} dA, & M_x &= \int_A y \sigma_{zz} dA \\
 U &= \int_A \bar{u} dA, & V &= \int_A \bar{v} dA, & W &= \int_A \bar{w} dA \\
 W_y &= \int_A x \bar{w} dA, & W_x &= \int_A y \bar{w} dA
 \end{aligned} \right\} (5)$$

$$\left. \begin{aligned}
 p_x &= \int_C \sigma_{xy} dx - \sigma_{xx} dy \\
 p_y &= \int_C \sigma_{yy} dx - \sigma_{xy} dy \\
 p_z &= \int_C \sigma_{yz} dx - \sigma_{xz} dy \\
 R_y &= \int_C x(\sigma_{yz} dx - \sigma_{xz} dy) \\
 R_x &= \int_C y(\sigma_{yz} dx - \sigma_{xz} dy)
 \end{aligned} \right\} (6)$$

The physical meaning of these quantities is clear:  $Q_x$  and  $Q_y$  are the shear forces and  $N$  is the tensile force in the beam;  $M_x$  and  $M_y$  are the usual bending moments;  $U$ ,  $V$ ,  $W$ ,  $W_x$ ,  $W_y$  are the displacement resultants and moments and will be discussed later;  $p_x$ ,  $p_y$ ,  $p_z$ , are the applied pressures, taken positive when acting in the direction opposite to the respective coordinate areas;  $R_x$  and  $R_y$  are the applied bending moments, taken positive in the aforementioned sense.

Equations (1) are now integrated over the cross section of the beam. The last of Eqs. (1) is multiplied by  $x$  and  $y$  respectively and integrated. The results are:

$$\left. \begin{aligned} Q_{x,z} - p_x &= \rho U_{,tt} \\ Q_{y,z} - p_y &= \rho V_{,tt} \\ N_{,z} - p_z &= \rho W_{,tt} \\ M_{y,z} - Q_x - R_y &= \rho W_{y,tt} \\ M_{x,z} - Q_y - R_x &= \rho W_{x,tt} \end{aligned} \right\} \quad (7)$$

The same procedure is now applied to the heat conduction equation, Eq. (2). Integrating Eq. (2) across the thickness, one has

$$\begin{aligned} k \int_A (\bar{T}_{,xx} + \bar{T}_{,yy}) dA + k \int_A \bar{T}_{,zz} dA - \rho c \int_A \bar{T}_{,t} dA = \\ (3\lambda + 2\mu) \alpha T_0 \int_A e_{,t} dA \end{aligned} \quad (8)$$

0

If Green's lemma is used and if it is assumed that differentiations with respect to  $z$  and  $t$  can be interchanged with the integration, one has

$$k T_{,zz} - \rho c T_{,t} + k \int_C (-\bar{T}_{,y} dx + \bar{T}_{,x} dy) = (3\lambda + 2\mu) \alpha T_0 \frac{\partial}{\partial t} \int_A e dA \quad (9)$$

where

$$T = \int_A \bar{T} dA \quad (10)$$

If the lateral surface of the beam is assumed to be insulated, then the integral  $\int_C (-\bar{T}_{,y} dx + \bar{T}_{,x} dy)$  vanishes since the integrand is proportional to  $\partial \bar{T} / \partial n$ . Then (9) becomes

$$k T_{,zz} - \rho c T_{,t} = (3\lambda + 2\mu) \alpha T_0 \frac{\partial}{\partial t} \int_A e dA \quad (11)$$

Multiplying Eq. (2) by  $x$  and  $y$  respectively and integrating yields

$$\left. \begin{aligned} k T_{y,zz} - \rho c T_{y,t} - k \int_A \bar{T}_{,x} dA &= (3\lambda + 2\mu) \alpha T_0 \frac{\partial}{\partial t} \int_A x e dA \\ k T_{x,zz} - \rho c T_{x,t} - k \int_A \bar{T}_{,y} dA &= (3\lambda + 2\mu) \alpha T_0 \frac{\partial}{\partial t} \int_A y e dA \end{aligned} \right\} \quad (12)$$

where

$$T_x = \int_A y \bar{T} dA \quad T_y = \int_A x \bar{T} dA \quad (13)$$

This is as far as one can go without assuming some sort of deformation and temperature variation across the thickness of the beam. For the deformations, it is not unreasonable to expect a linear variation; this will be taken to be

$$\left. \begin{aligned} \bar{u} &= u(z, t) \quad , \quad \bar{v} = v(z, t) \\ \bar{w} &= w(z, t) - xu_{,z} - yv_{,z} \end{aligned} \right\} \quad (14)$$

This assumption corresponds to Bernoulli-Euler bending theory. To include the effects of shear deformation, this assumption must be modified slightly; this point will be taken up in detail later. Since the outer surface of the beam is insulated, it would not be unreasonable to assume a linear variation in temperature as well. For other boundary conditions, such as a given temperature on the surface, this assumption would not necessarily be warranted.

With the assumption of linear temperature gradients, and using the definitions (10) and (13), one can express the temperature as

$$\begin{aligned} \bar{T} &= \frac{T(z, t)}{A} + x \frac{I_{xx} T_y(z, t) - I_{xy} T_x(z, t)}{I_{xx} I_{yy} - I_{xy}^2} \\ &+ y \frac{I_{yy} T_x(z, t) - I_{xy} T_y(z, t)}{I_{xx} I_{yy} - I_{xy}^2} \end{aligned} \quad (15)$$

This assumption gives rise to a fundamental inconsistency. The boundary condition for an insulated surface states that the normal derivative of the temperature vanishes on the lateral surface of the beam. Equation (15) will not, in general, satisfy this condition. It is felt, however, that this inconsistency is an approximation of accuracy comparable to the usual approximations of beam theory, and should yield a good degree of quantitative accuracy. A similar assumption was made by Biot<sup>1</sup> in a similar problem which was solved by a variational procedure.

The assumptions (14) imply that

$$\left. \begin{aligned} \sigma_{xx} &= \sigma_{yy} = 0 \\ \sigma_{xy} &= \sigma_{yz} = \sigma_{xz} = 0 \\ \sigma_{zz} &= E(w_{,z} - xu_{,zz} - yv_{,zz}) - Ea\bar{T} \\ e &= (1 - 2\nu)(w_{,z} - xu_{,zz} - yv_{,zz}) + (3\lambda + 2\mu) \frac{\alpha\bar{T}}{\lambda + \mu} \end{aligned} \right\} \quad (16)$$

These implications are inconsistent with the assumptions used in deriving Eqs. (7). These inconsistencies are inherent in beam theory, and the justification of the deformations as given by Eqs. (14) has been established by centuries of use.

The stress resultants become

$$\left. \begin{aligned} N &= EA w_{,z} - EaT \\ M_y &= -E(I_{yy}u_{,zz} + I_{xy}v_{,zz}) - EaT_y \\ M_x &= -E(I_{xy}u_{,zz} + I_{xx}v_{,zz}) - EaT_x \end{aligned} \right\} \quad (17)$$

From the third equation of (16), it is easily shown that

$$\begin{aligned} \sigma_{zz} = & \frac{N}{A} + x \frac{I_{xx} M_y - I_{xy} M_x}{I_{xx} I_{yy} - I_{xy}^2} + y \frac{I_{yy} M_x - I_{xy} M_y}{I_{xx} I_{yy} - I_{xy}^2} \\ & - E \alpha x \frac{I_{xx} T_y - I_{xy} T_x}{I_{xx} I_{yy} - I_{xy}^2} - E \alpha y \frac{I_{yy} T_x - I_{xy} T_y}{I_{xx} I_{yy} - I_{xy}^2} - E \alpha \frac{T}{A} \end{aligned} \quad (18)$$

Using the last of Eqs. (16) plus Eq. (15) to calculate  $\partial T / \partial y$  and  $\partial T / \partial x$ , one obtains

$$\left. \begin{aligned} k T_{,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{,t} = \\ (3\lambda + 2\mu) a T_0 (1 - 2\nu) A w_{,zt} \\ k T_{y,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu) a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{y,t} - k A \frac{I_{xx} T_y - I_{xy} T_x}{I_{xx} I_{yy} - I_{xy}^2} = \\ -(3\lambda + 2\mu) a T_0 (1 - 2\nu) (I_{yy} u_{,zzt} + I_{xy} v_{,zzt}) \\ k T_{x,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{x,t} - k A \frac{I_{yy} T_x - I_{xy} T_y}{I_{xx} I_{yy} - I_{xy}^2} = \\ -(3\lambda + 2\mu) a T_0 (1 - 2\nu) (I_{xy} u_{,zzt} + I_{xx} v_{,zzt}) \end{aligned} \right\} \quad (19)$$

It now remains only to apply assumptions (14) to the equations of motion, Eqs. (7). The result is

$$\left. \begin{aligned}
 Q_{x,z} - p_x &= \rho A u_{,tt} \\
 Q_{y,z} - p_y &= \rho A v_{,tt} \\
 N_{,z} - p_z &= \rho A w_{,tt} \\
 M_{y,z} - Q_x - R_y &= -\rho(I_{yy} u_{,ztt} + I_{xy} v_{,ztt}) \\
 M_{x,z} - Q_y - R_x &= -\rho(I_{xy} u_{,ztt} + I_{xx} v_{,ztt})
 \end{aligned} \right\} \quad (20)$$

The terms on the right-hand side of the last two equations of (20) are the rotatory inertia terms. Since the beam theory assumptions imply that the shear resultants  $Q_x$  and  $Q_y$  are zero, the usual procedure is to eliminate them from the first two of (20) by the last two equations. The result is

$$\left. \begin{aligned}
 M_{y,zz} - R_{y,z} - p_x &= \rho A u_{,tt} - \rho(I_{yy} u_{,zzt} + I_{xy} v_{,zzt}) \\
 M_{x,zz} - R_{x,z} - p_y &= \rho A v_{,tt} - \rho(I_{xy} u_{,zzt} + I_{xx} v_{,zzt}) \\
 N_{,z} - p_z &= \rho A w_{,tt}
 \end{aligned} \right\} \quad (21)$$

As a final exercise, the moment and axial force resultants are eliminated from Eqs. (21) by means of Eqs. (17), with the result

$$\begin{aligned}
 E a T_{y,zz} + E(I_{yy} u_{,zzz} + I_{xy} v_{,zzz}) + \rho A u_{,tt} \\
 - \rho(I_{yy} u_{,ztt} + I_{xy} v_{,ztt}) = -p_x \quad ; \quad (22)
 \end{aligned}$$

$$Ea T_{x,zz} + E(I_{xy} u_{,zzzz} + I_{xx} v_{,zzzz}) + \rho A v_{,tt} - \rho(I_{xy} u_{,zztt} + I_{xx} v_{,zztt}) = -p_y \quad ; \quad (23)$$

$$-Ea T_{,z} + EA w_{,zz} - \rho A w_{,tt} = p_z \quad (24)$$

The basic equations to be solved are, then, Eqs. (19), (22), (23), and (24). These constitute six equations in the six unknowns  $T$ ,  $T_x$ ,  $T_y$ ,  $u$ ,  $v$ ,  $w$ . Note that only the first of Eqs. (19) and Eq. (24) contain  $w$  and  $T$ ; thus, these two equations uncouple from the rest. When  $I_{xy} = 0$ , the other equations uncouple similarly into pairs.

### SECTION III

#### ANALYSIS OF TIMOSHENKO BEAM

In the previous section, the effects of rotatory inertia were included (hence the name "Rayleigh beam"<sup>9, 10</sup>). In this section, the effects of shear deformation will be included. The method of analysis will be quite analogous to that of the previous section.

The basic equations (1) through (4) obviously will be unchanged. The stress resultant, moment-resultant equations of motion, Eqs. (6), (7), and (8), will also be unchanged. The heat conduction equations (12) and (13) will be unchanged except in the term involving  $e$ .

To put in the shear deformation, one must change Eqs. (14). One assumes that

$$\left. \begin{aligned} \bar{u} &= u(z, t) \quad , \quad \bar{v} = v(z, t) \\ \bar{w} &= w(z, t) - x\phi_x(z, t) - y\phi_y(z, t) \end{aligned} \right\} \quad (25)$$

Here  $\phi_x$  and  $\phi_y$  represent the total rotation of the elements including the rotation due to bending and the rotation due to shear. The assumptions (25) imply that

$$\left. \begin{aligned} \sigma_{xx} &= \sigma_{yy} = 0 \\ \sigma_{zz} &= E(w_{,z} - x\phi_{x,z} - y\phi_{y,z}) - E\alpha T \\ e &= (1 - 2\nu)(w_{,z} - x\phi_{x,z} - y\phi_{y,z}) + (3\lambda + 2\mu) \frac{\alpha T}{\lambda + \mu} \end{aligned} \right\} \quad (26)$$

If, as before, one sets

$$T = \int_A \bar{T} dA, \quad T_x = \int_A y \bar{T} dA, \quad T_y = \int_A x \bar{T} dA, \quad (27)$$

then the assumption of a linear temperature profile implies that

$$\begin{aligned} \bar{T} = \frac{T}{A} + x \frac{I_{xx} T_y - I_{xy} T_x}{I_{xx} I_{yy} - I_{xy}^2} \\ + y \frac{I_{yy} T_x - I_{xy} T_y}{I_{xx} I_{yy} - I_{xy}^2} \end{aligned} \quad (15)$$

Using (15) and (26) in the heat conduction equations yields

$$\left. \begin{aligned} k T_{,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{,t} &= (3\lambda + 2\mu) a T_0 (1 - 2\nu) A w_{,zt} \\ k T_{y,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{y,t} - k A \frac{I_{xx} T_y - I_{xy} T_x}{I_{xx} I_{yy} - I_{xy}^2} &= \\ - (3\lambda + 2\mu) a T_0 (1 - 2\nu) (I_{yy} \phi_{x,t} + I_{xy} \phi_{y,t}) \\ k T_{x,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{(\lambda + \mu) \rho c} \right] T_{x,t} - k A \frac{I_{yy} T_x - I_{xy} T_y}{I_{xx} I_{yy} - I_{xy}^2} &= \\ - (3\lambda + 2\mu) a T_0 (1 - 2\nu) (I_{xy} \phi_{x,t} + I_{xx} \phi_{y,t}) \end{aligned} \right\} \quad (28)$$

The stress-resultant deformation equations are derived as before, except that the Timoshenko shear constants  $K_x$  and  $K_y$  multiply the expressions for  $Q_x$  and  $Q_y$ . (For a more complete discussion of these constants, see Ref. 11). The results are

$$\left. \begin{aligned} N &= EA w_{,z} - Ea T \\ M_y &= -E(I_{yy} \phi_{x,z} + I_{xy} \phi_{y,z}) - Ea T_y \\ M_x &= -E(I_{xy} \phi_{x,z} + I_{xx} \phi_{y,z}) - Ea T_x \\ Q_x &= K_x \mu A (u_{,z} - \phi_x) \\ Q_y &= K_y \mu A (v_{,z} - \phi_y) \end{aligned} \right\} \quad (29)$$

These expressions must now be substituted into the equations of motion (7); the result is

$$\left. \begin{aligned} K_x \mu A (u_{,zz} - \phi_{x,z}) - p_x &= \rho A u_{,tt} \\ K_y \mu A (v_{,zz} - \phi_{y,z}) - p_y &= \rho A v_{,tt} \\ EA w_{,zz} - Ea T_{,z} - p_z &= \rho A w_{,tt} \\ E(I_{yy} \phi_{x,zz} + I_{xy} \phi_{y,zz}) + K_x \mu A (u_{,z} - \phi_x) \\ &- \rho (I_{yy} \phi_{x,tt} + I_{xy} \phi_{y,tt}) + Ea T_{y,z} = -R_y \\ E(I_{xy} \phi_{x,zz} + I_{xx} \phi_{y,zz}) + K_y \mu A (v_{,z} - \phi_y) \\ &- \rho (I_{xy} \phi_{x,tt} + I_{xx} \phi_{y,tt}) + Ea T_{x,z} = -R_x \end{aligned} \right\} \quad (30)$$

Equations (28) and (30) constitute eight equations in the eight unknowns  $T$ ,  $T_x$ ,  $T_y$ ,  $u$ ,  $v$ ,  $w$ ,  $\phi_x$ ,  $\phi_y$ . The boundary conditions are the usual ones for the Timoshenko beam<sup>12</sup> plus those discussed in the previous section for the temperature terms.

## SECTION IV CALCULATIONS

The formulas derived herein will be used now to calculate the frequencies of the free oscillations of a simply supported beam for which  $I_{xy} = 0$ . (This is not necessarily a symmetric cross section.) Rotatory inertia and shear deformation will be neglected. Two cases will be discussed: axial vibrations with insulated ends, and bending vibrations with isothermal ends. Bending (and temperature variations) will be assumed to occur in the  $xz$  plane only.

With the above assumptions, the basic equations become -

$$\left. \begin{aligned} k T_{,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{\lambda + \mu} \right] T_{,t} &= \\ (3\lambda + 2\mu) a T_0 (1 - 2\nu) A w_{,zt} & \end{aligned} \right\} \quad (31)$$

$$\left. \begin{aligned} k T_{y,zz} - \rho c \left[ 1 - \frac{(3\lambda + 2\mu)^2 a^2 T_0}{\lambda + \mu} \right] T_{,t} - \frac{kA}{I_{yy}} T_y &= \\ -(3\lambda + 2\mu) a T_0 (1 - 2\nu) I_{yy} u_{,zzt} & \end{aligned} \right\}$$

$$\left. \begin{aligned} EI_{yy} u_{,zzzz} + \rho A u_{,tt} + E a T_{y,zz} &= 0 \\ EA w_{,zz} - \rho A w_{,tt} - E a T_{,z} &= 0 \end{aligned} \right\} \quad (32)$$

Thus it is seen that these equations uncouple into one pair involving  $w$  and  $T$  and another pair involving  $u$  and  $T_y$ . The equations involving  $w$  and  $T$  are the equations of axial vibration, and the equations involving  $u$  and  $T_y$  are the bending equations; they will be treated separately.

Case I: Axial vibrations, fixed and insulated ends

The boundary conditions are

$$w = T_{,z} = 0 \quad \text{at} \quad z = 0, l \quad (33)$$

The basic equations are

$$\left. \begin{aligned} T_{,zz} - \frac{\rho c}{k} (1 - \epsilon) T_{,t} - \frac{(3\lambda + 2\mu)\alpha T_0 (1 - 2\nu) A w_{,zt}}{k} &= 0 \\ w_{,zz} - \frac{1}{c_1^2} w_{,tt} - \frac{\alpha}{A} T_{,z} &= 0 \\ \epsilon &= \frac{(3\lambda + 2\mu)\alpha^2 T_0}{(\lambda + \mu)\rho c} \\ c_1^2 &= \frac{E}{\rho} \end{aligned} \right\} \quad (34)$$

It is easily seen that solutions satisfying the boundary condition (33) can be taken of the form

$$\left. \begin{aligned} w &= F(t) \sin \frac{n\pi z}{l} \\ T &= AG(t) \cos \frac{n\pi z}{l} \end{aligned} \right\} \quad (35)$$

0

Then Eqs. (34) become

$$\left. \begin{aligned} \frac{\rho c}{k} (1 - \epsilon) G + \frac{n^2 \pi^2}{l^2} G + \frac{n\pi}{l} \frac{(3\lambda + 2\mu) a T_0 (1 - 2\nu) \dot{F}}{k} &= 0 \\ \frac{1}{c_1} \ddot{F} + \frac{n^2 \pi^2}{l^2} F - \frac{an\pi}{l} G &= 0 \end{aligned} \right\} \quad (36)$$

Then solutions of the form  $\exp(i\omega t)$  exist providing  $\omega$  satisfies the equation

$$\left( \frac{-\omega^2}{c_1^2} - \frac{n^2 \pi^2}{l^2} \right) \left[ \frac{n^2 \pi^2}{l^2} + \frac{i\omega}{\kappa} (1 - \epsilon) \right] - \frac{n^2 \pi^2}{l^2} \frac{i\omega}{\kappa} \frac{\lambda + \mu}{3\lambda + 2\mu} \epsilon = 0 \quad (37)$$

where  $\kappa = k/\rho c$  is the thermal diffusivity. This is a cubic in  $\omega$ . If  $\epsilon = 0$  or is sufficiently small, there will be two roots close to  $\omega = n\pi c_1/l$ , which correspond to the free vibrations without thermal effects, and a root close to  $\omega = i(n^2 \pi^2/l^2)[\kappa/(1 - \epsilon)]$ , which corresponds to a decaying "thermal wave." It is of interest to compare the oscillations to those of a free vibration without thermal effects. One sets

$$\omega = \frac{n\pi c_1}{l} \zeta \quad \text{and} \quad \frac{l c_1}{\kappa n \pi} = \gamma \quad (38)$$

Thus (37) becomes

$$(\zeta^2 - 1)[1 + i\gamma(1 - \epsilon)\zeta] + i\gamma\epsilon \frac{\lambda + \mu}{3\lambda + 2\mu} \zeta = 0 \quad (39)$$

0

while  $\epsilon$  is usually small ( $\epsilon \approx 10^{-2}$ ),  $\gamma$  is large, at least for the smaller modes, ( $\gamma \approx 6 \times 10^7$ ). With this in mind, one finds that the roots close to  $\zeta = 1.0$  are given by

$$\zeta_1 = 1 + \frac{\epsilon}{2} \frac{\frac{1}{\gamma} + 1}{1/\gamma^2 + 1} \frac{\lambda + \mu}{3\lambda + 2\mu} + O(\epsilon^2) \quad (40)$$

$$\zeta_2 = -1 + \frac{\epsilon}{2} \frac{\frac{1}{\gamma} - 1}{1/\gamma^2 + 1} \frac{\lambda + \mu}{3\lambda + 2\mu} + O(\epsilon^2) \quad (41)$$

The root near  $\zeta = 1/\gamma$  is given by

$$\zeta_3 = \frac{1}{\gamma} + \frac{\epsilon}{\gamma} \frac{1/\gamma + 1}{1/\gamma^2 + 1} + O(\epsilon^2) \quad (42)$$

#### Case II: Bending vibration--pinned ends, ends kept at zero temperature

The basic equations are, from (31) and (32),

$$\left. \begin{aligned} T_{y,zz} - \frac{\rho c}{k} (1 - \epsilon) T_{y,t} - \frac{A}{I_{yy}} T_y + \frac{(3\lambda + 2\mu)\alpha T_0(1 - 2\nu)}{k} I_{yy} u_{,zzt} &= 0 \\ EI_{yy} u_{,zzzz} + \rho A u_{,tt} + E\alpha T_{y,zz} &= 0 \end{aligned} \right\} \quad (43)$$

It suffices to consider solutions of the form

$$\left. \begin{aligned} u &= F(t) \sin \frac{n\pi z}{l} \\ T_y &= G(t) \sin \frac{n\pi z}{l} \end{aligned} \right\} \quad (44)$$

Then

$$\frac{1-\epsilon}{\kappa} \dot{G} + \left( \frac{A}{I_{yy}} + \frac{n^2 \pi^2}{l^2} \right) G + \frac{(3\lambda + 2\mu)a T_0 (1 - 2\nu) I_{yy}}{\kappa} \frac{n^2 \pi^2}{l^2} \dot{F} = 0$$

$$\frac{\rho A}{EI_{yy}} \ddot{F} + \frac{n^4 \pi^4}{l^4} F - \frac{a}{I_{yy}} \frac{n^2 \pi^2}{l^2} G = 0$$

Solutions of the form  $\exp(i\omega t)$  exist if

$$\left( \omega^2 \frac{\rho A}{EI_{yy}} - \frac{n^4 \pi^4}{l^4} \right) \left[ \frac{A}{I_{yy}} + \frac{n^2 \pi^2}{l^2} + \frac{i\omega}{\kappa} (1 - \epsilon) \right] - \frac{i\omega}{\kappa} \frac{n^4 \pi^4}{l^4} \frac{\mu}{\lambda + 2\mu} = 0 \quad (45)$$

As before, there will be two solutions of the form

$$\omega = \pm \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI_{yy}}{\rho A}} \quad (46)$$

and one of the form

$$\omega = \frac{i\kappa}{1-\epsilon} \left( \frac{A}{I_{yy}} + \frac{n^2 \pi^2}{l^2} \right) \quad (47)$$

for sufficiently small  $\epsilon$ . We are mainly interested in the root given by Eq. (46). Accordingly, one sets

$$\left. \begin{aligned} \omega &= \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI_{yy}}{\rho A}} \zeta \\ \gamma &= \frac{n^2 \pi^2 / l^2}{A/I_{yy} + n^2 \pi^2 / l^2} \sqrt{\frac{EI_{yy}}{\rho A}} \frac{1}{\kappa} \end{aligned} \right\} \quad (48)$$

Equation (45) becomes

$$(\zeta^2 - 1)[1 + \epsilon \gamma (1 - \epsilon) \zeta] - \epsilon \gamma \frac{\mu}{\lambda + \mu} \zeta = 0 \quad (49)$$

This is an equation that is completely analogous to Eqs. (34) of the last section, except that  $\gamma$  is given by Eq. (48) instead of (38) and one must replace  $\lambda + \mu/(3\lambda + 2\mu)$  by  $\mu/(\lambda + \mu)$ . The roots are given by

$$\left. \begin{aligned} \zeta_1 &= 1 + \frac{\epsilon}{2} \frac{1 + \epsilon/\gamma}{1 + 1/\gamma^2} \frac{\mu}{\lambda + \mu} + O(\epsilon^2) \\ \zeta_2 &= -1 + \frac{\epsilon}{2} \frac{-1 + \epsilon/\gamma}{1 + 1/\gamma^2} \frac{\mu}{\lambda + \mu} + O(\epsilon^2) \\ \zeta_3 &= \frac{\epsilon}{\gamma} + \frac{\epsilon^2}{\gamma} \frac{1 + 1/\gamma}{1 + 1/\gamma^2} + O(\epsilon^2) \end{aligned} \right\} \quad (50)$$

be it temperature, deformation, or stress, and then to integrate this distribution across the cross section to obtain resultants. These resultants are then the quantities which are considered to be the dependent variables of the resulting differential equations. Evidently, the success of the method depends upon how closely the assumed distribution approximates the actual distribution. In the case of thin beams, plates, and shells, a linear distribution suffices, for all practical cases, to describe the stresses and deformations. For temperature distributions, the situation is different. Depending on the problem, a linear distribution might or might not be indicated. In the case of a plate or shell, this is not unreasonable; in the case of a beam, the success of the linear approximation definitely would depend upon the individual problem at hand. The insulated beam was chosen for two reasons: First, it is believed that the linear distribution is probably a good approximation here since there are no prescribed temperature gradients over the cross section. Second, it is a natural boundary condition since such quantities as  $\int_C \partial T / \partial n \, ds$  vanish for an insulated lateral surface. To extend beam theory to include the other types of boundary conditions, one must use some further approximations to obtain the boundary integrals occurring in the averaged heat conduction equation.

The effects of shear deformation and rotatory inertia are definitely of second order for most practical cases, becoming of importance mainly at high frequencies, or where the shape of the cross section changes abruptly or rapidly. The Biot coupling is also a small effect,<sup>2</sup> becoming of importance mainly when the dilatation changes rapidly. This would more or less imply that for dynamic problems both effects would come into play under similar circumstances, and it would be prudent to include all three effects in any analysis where it is thought that any one effect would come into play. (The case of a short beam in static loading where shear deformation is of importance is an obvious exception to the above statement.)

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UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. THERMOELASTIC VIBRATIONS OF A BEAM, prepared by J. P. Jones, 20 February 1963. [28]p. incl. illus. (Report TDR-169(3153-06)TR-1; BSD-TDR-63-17) (Contract AF 04(695)-169) Unclassified report</p> <p>The equations of motion of a beam with the lateral surface thermally insulated are derived, including the effects of shear deformation and rotary inertia. Thermoelectric coupling in the heat conduction equation and in the elastic constitutive relations is also included. Axial deformations of the beam are taken into account. Two examples are presented and discussed.</p>
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